NEAR MISS abc-TRIPLES IN COMPACTLY BOUNDED SUBSETS

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Abstract. In the present paper, we study the existence of near miss abc-triples in compactly bounded subsets. In more concrete terms, we prove that there exist infinitely many abc-triples such that:

(1) $|abc|$ exceeds a certain quantity determined by the product of the distinct prime numbers of abc, and, moreover,
(2) a certain value λ determined by a, b, c, which corresponds to the quantity "λ" in the Legendre form of an elliptic curve, lies in a given compactly bounded subset.

0. **Introduction**

First, we review the definition of an abc-triple (cf. Definition 1.5).

Definition 0.1. Let $a, b, c \in \mathbb{Z}$ be such that

$$a + b + c = 0,$$

$$(a, b) = 1,$$

$$a \neq 0,\ b \neq 0,\ c \neq 0.$$

Then we shall say that the triad of integers (a, b, c) is an abc-triple. For an abc-triple (a, b, c), we define

$$N_{(a,b,c)} := \prod_{\substack{p \in \mathfrak{Primes} \\ p | abc}} p, \qquad \lambda_{(a,b,c)} := -\tfrac{b}{a}.$$

Next, we state the abc Conjecture.

Theorem 0.2 (abc Conjecture)**.** *For $\gamma \in \mathbb{R}_{>0}$, there exists a $C_\gamma \in \mathbb{R}_{>0}$ such that, for every abc-triple (a, b, c), the following inequality holds:*

$$\max\{|a|, |b|, |c|\} < C_\gamma N_{(a,b,c)}^{1+\gamma}.$$

In 1988, Masser proved that the $\gamma = 0$ version of the abc Conjecture does not hold. The result obtained by Masser (cf. [M], Theorem) is as follows:

Theorem 0.3. *Let $N_0, \gamma \in \mathbb{R}_{>0}$ be such that $\gamma < \frac{1}{2}$. Then there exists an abc-triple (a, b, c) such that*

(Masser 1) $$N_{(a,b,c)} > N_0,$$

(Masser 2) $$|abc| > N_{(a,b,c)}^3 \exp\left((\log N_{(a,b,c)})^{\frac{1}{2} - \gamma} \right).$$

Since any infinite collection of *abc*-triples as in Theorem 0.3 for $N_0 \to +\infty$ yields a counterexample to the $\gamma = 0$ version of the *abc* Conjecture, we shall refer to such *abc*-triples as *near miss abc-triples*.

On the other hand, in [GenEll], Mochizuki introduced the notion of a *compactly bounded subset* (cf. [GenEll], Example 1.3, (ii)) and showed that the *abc* Conjecture holds for *arbitrary abc*-triples if and only if it holds for *abc*-triples that lie (i.e., for which the associated "$\lambda_{(a,b,c)}$" lies) in a *given compactly bounded subset* (cf. [GenEll], Theorem 2.1). Before proceeding, we review the definition of a compactly bounded subset (cf. Definition 1.6).

Definition 0.4. Let $r \in \mathbb{Q}$, $\varepsilon \in \mathbb{R}_{>0}$, and Σ a finite subset of the set of valuations on $\mathbb{Q}$ which includes the unique archimedean valuation ∞ on $\mathbb{Q}$. Write

$$K_{r,\varepsilon,\Sigma} := \{r' \in \mathbb{Q} \mid \|r' - r\|_v \leq \varepsilon,\ \forall v \in \Sigma\}.$$

We shall refer to $K_{r,\varepsilon,\Sigma}$ as an (r, ε, Σ)-*compactly bounded subset.*

(Here, we remark that the use of the indefinite article "an" preceding the expression "(r, ε, Σ)-compactly bounded subset" results from the usage of this expression in [GenEll], where one considers compactly bounded subsets of more general hyperbolic curves than just the projective line minus three points (which corresponds to the situation considered in the present paper) over more general number fields than just $\mathbb{Q}$.)

In the present paper, we prove that the existence of near miss *abc*-triples that lie in a given compactly bounded subset. The main result of the present paper is as follows:

Theorem 0.5. *Let $r \in \mathbb{Q}$; $\varepsilon, N_0, \gamma \in \mathbb{R}_{>0}$ such that $\gamma < \frac{1}{2}$; Σ a finite subset of the set of valuations on $\mathbb{Q}$ which includes the unique archimedean valuation ∞ on $\mathbb{Q}$; and $K_{r,\varepsilon,\Sigma}$ an (r, ε, Σ)-compactly bounded subset. Then there exists an abc-triple (a, b, c) such that*

$$N_{(a,b,c)} > N_0, \tag{Main 1}$$

$$|abc| > N^3_{(a,b,c)} \exp\left((\log\log N_{(a,b,c)})^{\frac{1}{2}-\gamma}\right), \tag{Main 2}$$

$$\lambda_{(a,b,c)} \in K_{r,\varepsilon,\Sigma}. \tag{Main 3}$$

In §1, we establish the notation and terminology used in the present paper. In §2, we review the statement of Theorem 0.5 (cf. Theorem 2.1) and state the elliptic curve version of Theorem 0.5 (cf. Theorem 2.7). Also, we discuss a certain related conjecture. In §3, we review well-known consequences of the Prime Number Theorem. One such consequence is Theorem 3.9, which estimates the cardinality of the set

$$\{x' \in \mathbb{Z}_{>0} \mid 2 \leq x' \leq x,\ \mathrm{LPN}(x') \leq y,\ \text{and } (x', n) = 1\},$$

where $\mathrm{LPN}(-)$ denotes the largest prime number dividing the integer in parentheses. This estimate plays an important role in §4. In §4, we prove Theorem 0.5 (i.e. Theorem 2.1). In §5, we give, for the convenience of the reader, an exposition of the proof of Masser's result, i.e., Theorem 0.3, via arguments similar to the arguments given in the proof of Theorem 0.5 in §4. For instance, Lemmas 5.1 and 5.2 correspond to Lemmas 4.1 and 4.4, respectively.

The proof of Theorem 0.5 is divided into Lemmas 4.1, 4.2, 4.3, and 4.4. Lemmas 4.1 and 4.4 are based on the arguments of Masser's proof. In particular, by applying

Lemma 4.1 (which corresponds to Lemma 5.1), we obtain an abc-triple that can in fact be shown (i.e., by applying the arguments of Lemma 4.4 or Lemma 5.2) to satisfy the conditions (Masser 1) and (Masser 2) of Theorem 0.3, but whose associated "λ" is *not necessarily contained in the compactly bounded subset* $K_{r,\varepsilon,\Sigma}$ *of condition (Main 3).* This state of affairs is remedied as follows:

- First, we apply Lemma 4.1 to construct a pair of integers (a_1, b_1) which satisfies the conditions (Masser 1) and (Masser 2) of Theorem 0.3, and whose associated "λ" is contained in a $(1, \varepsilon, \Sigma)$-compactly bounded subset.
- Next, we apply Lemma 4.2 to construct a pair of integers (a_2, b_2) (which does *not necessarily* satisfy the conditions (Masser 1) and (Masser 2) of Theorem 0.3, but) whose associated "λ" is contained in an $(r, \varepsilon, \Sigma \setminus \{\infty\})$-compactly bounded subset.
- Lemma 4.3 is the *key step* in the proof of Theorem 0.5 and may be summarized as follows: It follows immediately from the inequalities
$$1 < \tfrac{b_1}{a_1} < \tfrac{r+\varepsilon}{r-\varepsilon}$$
(which are an immediate consequence of the construction of (a_1, b_1) in Lemma 4.1), by considering the elementary geometry of the real line, that there exists an $\alpha' \in \mathbb{Z}$ such that
$$\| \tfrac{b_2}{a_2} \left(\tfrac{b_1}{a_1}\right)^{\alpha'} - r\|_\infty \leq \varepsilon.$$
We define (a_3, b_3) to be the unique pair of relatively prime positive integers such that
$$\tfrac{b_3}{a_3} := \tfrac{b_2}{a_2} \left(\tfrac{b_1}{a_1}\right)^{\alpha'}.$$
Then it follows formally from the defining property of a non-archimedean valuation that the "λ" associated to the pair of integers (a_3, b_3) is contained in an (r, ε, Σ)-compactly bounded subset.
- Finally, in Lemma 4.4, we estimate the quantity $N_{(a,b,c)}$ associated to the abc-triple $(a := a_3, b := -b_3, c := -a - b)$ and thus conclude that this abc-triple (a, b, c) satisfies the conditions (Main 1), (Main 2), and (Main 3) of Theorem 0.5.

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1. **Notation**

Elementary Notation

Here, we introduce some elementary notation.

Definition 1.1. Let X be a finite set. Then we shall write $\sharp X$ for the cardinality of X.

Definition 1.2.

(1) Write $\mathbb{Z}$ for the ring of rational integers, $\mathbb{Q}$ for the field of rational numbers, $\mathbb{R}$ for the field of real numbers, and $\mathbb{C}$ for the field of complex numbers.
(2) Let $\Lambda \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$ and $a \in \Lambda$. Then we define
$$\Lambda_{>a} := \{a' \in \Lambda \mid a' > a\},\ \Lambda_{\geq a} := \{a' \in \Lambda \mid a' \geq a\}.$$
(3) Let $m, n \in \mathbb{Z} \setminus \{0\}$. Then we shall write (m, n) for the greatest common divisor of $|m|$ and $|n|$.

Definition 1.3.

(1) Write $\mathfrak{Primes}$ for the set of prime numbers.
(2) Write $\mathbb{V}$ for the set of (archimedean and non-archimedean) valuations on $\mathbb{Q}$. We denote the unique archimedean valuation on $\mathbb{Q}$ by ∞. Write $\mathbb{V}^{\mathrm{arc}} := \{\infty\}$, $\mathbb{V}^{\mathrm{non}} := \mathbb{V} \backslash \{\infty\}$. Here, we suppose that $\|-\|_v$ is normalized as follows: $\|\lambda\|_v = |\lambda|$ for $\lambda \in \mathbb{Q}$ if $v \in \mathbb{V}^{\mathrm{arc}}$; there exists a (unique) $p_v \in \mathfrak{Primes}$ such that $\|p_v\|_v = p_v^{-1}$ if $v \in \mathbb{V}^{\mathrm{non}}$.
(3) For $p \in \mathfrak{Primes}$, write $\mathbb{Z}_p$ for the ring of p-adic integers and $\mathbb{Q}_p$ for the field of p-adic numbers.

Definition 1.4.

(1) Let X be a set and $f, g \colon X \to \mathbb{C}$. We shall write
$$f = O(g)$$
if there exists an $M \in \mathbb{R}_{>0}$ such that, for every $x \in X$,
$$|f(x)| \leq M|g(x)|.$$
We shall also write $f(x) = O(g(x))$ instead of $f = O(g)$.
(2) Let X, Y be sets, U a subset of $X \times Y$, and $f, g \colon U \to \mathbb{C}$. We shall write
$$f = O_Y(g)$$
if there exists an $M_Y \colon Y \to \mathbb{R}_{>0}$ such that, for every $(x, y) \in U$,
$$|f(x, y)| \leq M_Y(y)|g(x, y)|.$$
We shall also write $f(x, y) = O_Y(g(x, y))$, $f(x, y) = O_y(g(x, y))$, or $f = O_y(g)$ instead of $f = O_Y(g)$.

abc-Triples and Compactly Bounded Subsets

Here, we define *abc*-triples and compactly bounded subsets, which play an essential role in the present paper.

Definition 1.5. Let $a, b, c \in \mathbb{Z}$ be such that
$$a + b + c = 0,$$
$$(a, b) = 1,$$
$$a \neq 0,\ b \neq 0,\ c \neq 0.$$

Then we shall say that the triad of integers (a, b, c) is an abc-triple. For an abc-triple (a, b, c), we define

$$N_{(a,b,c)} := \prod_{\substack{p \in \mathfrak{Primes} \\ p \mid abc}} p, \qquad \lambda_{(a,b,c)} := -\tfrac{b}{a}.$$

Definition 1.6. Let $r \in \mathbb{Q}$, $\varepsilon \in \mathbb{R}_{>0}$, and $\Sigma \subseteq \mathbb{V}$ a finite subset which includes ∞. Write

$$K_{r,\varepsilon,\Sigma} := \{r' \in \mathbb{Q} \mid \|r' - r\|_v \leq \varepsilon, \, \forall v \in \Sigma\}.$$

We shall refer to $K_{r,\varepsilon,\Sigma}$ as an *(r, ε, Σ)-compactly bounded subset.*

Definitions Related to Prime Numbers

Here, we define various definitions related to prime numbers.

Definition 1.7. Let $i \in \mathbb{Z}_{\geq 1}$, $n \in \mathbb{Z} \setminus \{0\}$, $x, y \in \mathbb{R}_{>0}$.

(1) We denote the i-th smallest prime number by p_i.
(2) If $n \neq \pm 1$, then we denote the largest prime number dividing n by $\mathrm{LPN}(n)$. If $n = \pm 1$, then we set $\mathrm{LPN}(n) := 1$.
(3) We define
$$\pi(x) := \sharp\{x' \in \mathfrak{Primes} \mid x' \leq x\}.$$
(4) We define
$$\Psi(x, y) := \sharp\{x' \in \mathbb{Z} \mid 2 \leq x' \leq x, \, \mathrm{LPN}(x') \leq y\}.$$
(5) We define
$$\Psi(x, y; n) := \sharp\{x' \in \mathbb{Z} \mid 2 \leq x' \leq x, \, \mathrm{LPN}(x') \leq y, \, (x', n) = 1\}.$$
(6) We define
$$\theta(x) := \sum_{\mathfrak{Primes} \ni p \leq x} \log p.$$

Facts Related to Elliptic Curves

Here, we review facts related to elliptic curves.

Definition 1.8.

(1) Write $\mathbb{G}_{\mathrm{m}} := \mathrm{Spec}\ \mathbb{Z}[T, T^{-1}]$ for the multiplicative group scheme over $\mathbb{Z}$ and $\mathbb{G}_{\mathrm{a}} := \mathrm{Spec}\ \mathbb{Z}[T]$ for the additive group scheme over $\mathbb{Z}$.
(2) Let k be a field. We shall say that E is an *elliptic curve over k* if E is an irreducible smooth projective curve over k, $\dim_k \Gamma\left(E, \omega_{E/k}\right) = 1$, and there exists a k-morphism $e\colon \mathrm{Spec}\ k \to E$.

Definition 1.9. Let us consider the equation

$$\mathbb{E}\colon y^2 = x^3 + a_1x^2 + a_2x + a_3 \text{ for } a_1, a_2, a_3 \in \mathbb{Q}.$$

We define the *discriminant* $D_{\mathbb{E}}$ of $\mathbb{E}$ to be the discriminant of the cubic equation $x^3 + a_1x^2 + a_2x + a_3$. Note that $\mathbb{E}$ defines an elliptic curve E over $\mathbb{Q}$ if $D_{\mathbb{E}} \neq 0$.

Remark 1.10. Let $\mathbb{E}$ be as in Definition 1.9. Note that $D_{\mathbb{E}}$ *differs* from the quantity "Δ" that is referred to as the "discriminant" in [S], III.1. According to [S], III.1, it holds that $\Delta = 2^4 D_{\mathbb{E}}$.

Remark 1.11. Let E be an elliptic curve over $\mathbb{Q}$. Then, in general, an equation "$\mathbb{E}$" as in Definition 1.9 that gives rise to E is *not uniquely determined*. In particular, it does not make sense to speak of the "discriminant D_E associated to E". On the other hand, it does make sense to speak of the *minimal discriminant* associated to E, as defined in [S], VIII.8. We shall write $D_E^{\min}$ for the minimal discriminant associated to E.

Definition 1.12. Let $p \in \mathfrak{Primes}$ and $\kappa := \mathbb{Z}_p/p\mathbb{Z}_p$.

(1) We shall say that E has *good reduction at* p if there exists a smooth projective $\mathbb{Z}_p$-scheme E' such that $E' \times_{\mathbb{Z}_p} \mathbb{Q}_p$ and $E \times_{\mathbb{Q}} \mathbb{Q}_p$ are isomorphic as $\mathbb{Q}_p$-schemes.
(2) We shall say that E has *multiplicative reduction at* p if there exists a smooth group scheme E' over $\mathbb{Z}_p$ such that $E' \times_{\mathbb{Z}_p} \mathbb{Q}_p$ is isomorphic to $E \times_{\mathbb{Q}} \mathbb{Q}_p$ as a group scheme over $\mathbb{Q}_p$, and $E' \times_{\mathbb{Z}_p} \kappa$ is isomorphic to $\mathbb{G}_{\mathrm{m}}$ as a group scheme over some algebraic closure of κ.
(3) We shall say that E has *additive reduction at* p if there exists a smooth group scheme E' over $\mathbb{Z}_p$ such that $E' \times_{\mathbb{Z}_p} \mathbb{Q}_p$ is isomorphic to $E \times_{\mathbb{Q}} \mathbb{Q}_p$ as a group scheme over $\mathbb{Q}_p$, and $E' \times_{\mathbb{Z}_p} \kappa$ is isomorphic to $\mathbb{G}_{\mathrm{a}}$ as a group scheme over some algebraic closure of κ.
(4) We define the *conductor* N_E of E (cf. Remark 1.13) to be the product

$$N_E := \prod_{p \in \mathfrak{Primes}} p^{f_p(E)},$$

where $f_p(E) := 0$ if E has good reduction at p; $f_p(E) := 1$ if E has multiplicative reduction at p; and $f_p(E) := 2$ if E has additive reduction at p.

Remark 1.13. The definition of the conductor given in Definition 1.12 is not quite correct, but suffices for the purposes of the present paper. For a more detailed discussion of this "incorrect working definition", we refer to [S], VIII. 11.

2. **The Main Result**

The following theorem is the main result of the present paper. The proof of this result is given in §4.

Theorem 2.1. *Let $r \in \mathbb{Q}$; $\varepsilon, N_0, \gamma \in \mathbb{R}_{>0}$ such that $\gamma < \frac{1}{2}$; $\Sigma \subseteq \mathbb{V}$ a finite subset which includes ∞; and $K_{r,\varepsilon,\Sigma}$ an (r, ε, Σ)-compactly bounded subset (cf. Definition 1.6). Then there exists an abc-triple (a, b, c) such that*

$$N_{(a,b,c)} > N_0, \tag{Main 1}$$

$$|abc| > N_{(a,b,c)}^3 \exp\left((\log\log N_{(a,b,c)})^{\frac{1}{2}-\gamma}\right), \tag{Main 2}$$

$$\lambda_{(a,b,c)} \in K_{r,\varepsilon,\Sigma}. \tag{Main 3}$$

For the sake of comparison, we also state Masser's result. Masser's proof of this result is reviewed in §5.

Theorem 2.2. *Let $N_0, \gamma \in \mathbb{R}_{>0}$ be such that $\gamma < \frac{1}{2}$. Then there exists an abc-triple (a, b, c) such that*

$$N_{(a,b,c)} > N_0, \qquad \text{(Masser 1)}$$

$$|abc| > N_{(a,b,c)}^3 \exp\left((\log N_{(a,b,c)})^{\frac{1}{2}-\gamma}\right). \qquad \text{(Masser 2)}$$

Our result is motivated by Masser's. Unlike the *abc*-triple (a, b, c) of Theorem 2.2, the *abc*-triple (a, b, c) of Theorem 2.1 is subject to the condition that it *lie inside an* (r, ε, Σ)*-compactly bounded subset* (i.e., (Main 3)); on the other hand, the inequality of Theorem 2.1 (i.e., (Main 2)) is *weaker* than the inequality of Theorem 2.2 (i.e., (Masser 2)).

Theorem 2.1 may be translated into the language of algebraic geometry (cf. Theorem 2.7 below), by applying the so-called Frey Curve, which we review in the following lemma.

Lemma 2.3. *Let (a, b, c) be an abc-triple. Thus, the equation*

$$\mathbb{E}\colon y^2 = x(x + a)(x - b)$$

defines an elliptic curve E over $\mathbb{Q}$. Then there exists $e \in \{0, 1\}$ such that

$$|D_{\mathbb{E}}| = |abc|^2, \; N_E = 2^e N_{(a,b,c)}.$$

Proof. It follows from the definition of $D_{\mathbb{E}}$ that

$$|D_{\mathbb{E}}| = |abc|^2.$$

It follows from [S], Chapter VIII, Lemma 11.3 (b) (and its proof) that

$$N_E = 2^e N_{(a,b,c)}.$$

This completes the proof. □

Remark 2.4. According to [S], Chapter VIII, Lemma 11.3 (a), it follows that there exists an $e' \in \{0, 1\}$ such that

$$D_E^{\min} = 2^{4-12e'}|abc|^2 = 2^{4-12e'}|D_{\mathbb{E}}|,$$

where $D_E^{\min}$ is the minimal discriminant associated to E (cf. Remark 1.11).

Before mentioning the elliptic curve version of Theorem 2.1, we review the statement of (a weakened version of) the Szpiro Conjecture (cf. [IUTchIV], Theorem A), which played an important role in motivating both [M] and the present paper.

Theorem 2.5 (Szpiro Conjecture)**.** *Let $\delta \in \mathbb{R}_{>0}$. Then there exists a $C_\delta \in \mathbb{R}_{>0}$ such that, for every equation $\mathbb{E}$ as in Lemma 2.3, the following inequality holds:*

$$|D_{\mathbb{E}}| \leq C_\delta N_E^{6+\delta}.$$

Remark 2.6. The original Szpiro Conjecture is as follows:

> *Let $\delta \in \mathbb{R}_{>0}$. Then there exists a $C_\delta \in \mathbb{R}_{>0}$ such that, for every elliptic curve E over $\mathbb{Q}$, the following inequality holds:*
>
> $$D_E^{\min} \leq C_\delta N_E^{6+\delta},$$
>
> *where $D_E^{\min}$ is the minimal discriminant associated to E (cf. Remark 1.11).*

It follows immediately from the above statement and Remark 2.4 that Theorem 2.5 is equivalent to the original Szpiro Conjecture.

By Lemma 2.3, we obtain the following elliptic curve version of Theorem 2.1.

Theorem 2.7. *Let $r \in \mathbb{Q}$, $\varepsilon \in \mathbb{R}_{>0}$, $\Sigma \subseteq \mathbb{V}$ a finite subset which includes ∞, $K_{r,\varepsilon,\Sigma}$ an (r, ε, Σ)-compactly bounded subset (cf. Definition 1.6), and*

$$\mathscr{M}_{r,\varepsilon,\Sigma} := \{\mathbb{E} \,|\mathbb{E}\colon y^2 = x(x+a)(x-b) \text{ for an abc-triple } (a,b,c) \text{ s.t. } \lambda_{(a,b,c)} \in K_{r,\varepsilon,\Sigma}\}.$$

Then, for $N_0, \gamma \in \mathbb{R}_{>0}$ such that $\gamma < \frac{1}{2}$, there exist infinitely many equations $\mathbb{E} \in \mathscr{M}_{r,\varepsilon,\Sigma}$ such that

$$N_E > N_0,\ |D_{\mathbb{E}}| > N_E^6 \exp\left((\log\log N_E)^{\frac{1}{2}-\gamma}\right).$$

Remark 2.8. Note that $\lambda_{(a,b,c)}$ may be regarded as the quantity "λ" that appears in the Legendre form of the corresponding elliptic curve. In particular, even on $\mathscr{M}_{r,\varepsilon,\Sigma}$, if one takes the "$\delta$" of Theorem 2.5 to be 0, then the resulting inequality does not hold. Note that Theorem 2.2 implies that, if one takes the "δ" of Theorem 2.5 to be 0, then the resulting inequality does not hold.

Finally, we remark that Theorem 2.1 may be regarded as a weakened version of the following conjecture, which was motivated by the theory of [IUTchIV], §1, §2. This conjecture may be understood as a conjecture to the effect that a version of Masser's result (i.e., Theorem 2.2) holds, *even when the abc-triple is subject to the further condition that it lie in a given (r, ε, Σ)-compactly bounded subset $K_{r,\varepsilon,\Sigma}$*.

Conjecture 2.9. *Let $r \in \mathbb{Q}$; $\varepsilon, N_0, \gamma \in \mathbb{R}_{>0}$; $\Sigma \subseteq \mathbb{V}$ a finite subset which includes ∞; and $K_{r,\varepsilon,\Sigma}$ an (r, ε, Σ)-compactly bounded subset (cf. Definition 1.6) such that $\gamma < \frac{1}{2}$. Then there exists an abc-triple (a, b, c) such that*

$$N_{(a,b,c)} > N_0,$$
$$|abc| > N_{(a,b,c)}^3 \exp\left((\log N_{(a,b,c)})^{\frac{1}{2}-\gamma}\right),$$
$$\lambda_{(a,b,c)} \in K_{r,\varepsilon,\Sigma}.$$

3. Review of Well-Known Consequences of the Prime Number Theorem

We shall use (the version that includes the error term of) the Prime Number Theorem without proof. A proof may be found in [T], II.4.1, Theorem 1.

Theorem 3.1 (Prime Number Theorem)**.** *Let $x \in \mathbb{R}_{\geq 2}$. Then there exists a $C \in \mathbb{R}_{>0}$ such that the following estimate holds:*

$$\pi(x) = \mathrm{li}(x) + O\left(x \exp(-C(\log x)^{\frac{1}{2}})\right),$$

where we write

$$\mathrm{li}(x) := \int_2^x \tfrac{1}{\log t} dt.$$

Before stating various consequences of Theorem 3.1, we prove the following lemma.

Lemma 3.2. *Let $x \in \mathbb{R}_{\geq 2}$, $n \in \mathbb{Z}_{\geq 1}$. Then the following estimate holds:*

$$\int_2^x \tfrac{1}{(\log t)^n} dt = O_n\left(\tfrac{x}{(\log x)^n}\right) = \tfrac{x}{(\log x)^n} + O_n\left(\tfrac{x}{(\log x)^{n+1}}\right).$$

Proof. Write

$$f(x) := -\frac{2x}{(\log x)^n} + \int_2^x \frac{1}{(\log t)^n} dt \text{ for } x \geq 2.$$

Since

$$f'(x) = -\frac{1}{(\log x)^n} + \frac{2n}{(\log x)^{n+1}} = -\frac{1}{(\log x)^n}\left(1 - \frac{2n}{\log x}\right)$$

is < 0 for x sufficiently large, it follows that there exists an $M_n \in \mathbb{R}_{>0}$ such that

$$-\frac{2x}{(\log x)^n} + \int_2^x \frac{1}{(\log t)^n} dt = f(x) \leq M_n.$$

Thus, it follows that

$$0 \leq \int_2^x \frac{1}{(\log t)^n} dt \leq \frac{2x}{(\log x)^n} + M_n.$$

Since $\frac{x}{(\log x)^n} \to +\infty$ as $x \to +\infty$, it follows that

$$\int_2^x \frac{1}{(\log t)^n} dt = O_n\left(\frac{x}{(\log x)^n}\right).$$

By applying partial integration and the above estimate, it follows that

$$\int_2^x \frac{1}{(\log t)^n} dt = \frac{x}{(\log x)^n} - \frac{2}{(\log 2)^n} + n\int_2^x \frac{1}{(\log t)^{n+1}} dt = \frac{x}{(\log x)^n} + O_n\left(\frac{x}{(\log x)^{n+1}}\right).$$

This completes the proof. □

Theorem 3.1 and Lemma 3.2 easily implies the following two corollaries.

Corollary 3.3. *Let $x \in \mathbb{R}_{\geq 2}$. Then the following estimate holds:*

$$\pi(x) = \frac{x}{\log x} + \frac{x}{(\log x)^2} + O\left(\frac{x}{(\log x)^3}\right).$$

Proof. First, it follows from Theorem 3.1 that there exists a $C \in \mathbb{R}_{>0}$ such that the following estimate holds:

$$\pi(x) = \mathrm{li}(x) + O\left(x \exp(-C(\log x)^{\frac{1}{2}})\right).$$

Next, by applying partial integration to $\mathrm{li}(x)$, it follows from Lemma 3.2 that

$$\begin{aligned}\mathrm{li}(x) &= \int_2^x \frac{1}{\log t} dt \\ &= \frac{x}{\log x} - \frac{2}{\log 2} + \int_2^x \frac{1}{(\log t)^2} dt \\ &= \frac{x}{\log x} - \frac{2}{\log 2} + \frac{x}{(\log x)^2} - \frac{2}{(\log 2)^2} + \int_2^x \frac{2}{(\log t)^3} dt \\ &= \frac{x}{\log x} + \frac{x}{(\log x)^2} + O\left(\frac{x}{(\log x)^3}\right).\end{aligned}$$

Finally, it follows from an elementary calculation that

$$\exp(-C(\log x)^{\frac{1}{2}}) = O\left(\frac{1}{(\log x)^3}\right).$$

Thus, it follows that

$$\pi(x) = \frac{x}{\log x} + \frac{x}{(\log x)^2} + O\left(\frac{x}{(\log x)^3}\right).$$

This completes the proof. □

Corollary 3.4. *Let $x \in \mathbb{R}_{\geq 2}$. Then the following estimate holds:*

$$\theta(x) = x + O\left(\tfrac{x}{(\log x)^2}\right).$$

Proof. The estimate in question may be obtained by computing Lebesgue-Stieltjes integrals, applying Lemma 3.2 as follows:

$$\begin{aligned}
\sum_{i=1}^{\pi(x)} \log p_i &= \int_{2-0}^{x} \log t \; d\pi(t) \\
&= \pi(x)\log x - \int_{2-0}^{x} \tfrac{\pi(t)}{t} dt \\
&= x + \tfrac{x}{\log x} + O\left(\tfrac{x}{(\log x)^2}\right) - \int_{2-0}^{x} \left(\tfrac{1}{\log t} + O\left(\tfrac{1}{(\log t)^2}\right)\right) dt \\
&= x + O\left(\tfrac{x}{(\log x)^2}\right).
\end{aligned}$$

Here, we note that the estimate of the third equality follows by applying the estimate of Corollary 3.3. $\square$

In order to prove Theorem 2.1, it will be necessary to apply certain estimates concerning Ψ functions. The various estimates concerning Ψ functions discussed in the remainder of the present §3 involve a real number "y" that satisfies only rather weak hypotheses. In fact, in the proof of the main results of the present paper in §4, it will only be necessary to apply these estimates in the case where $y = (\log x)^{\frac{1}{2}}$. On the other hand, we present these estimates for more general "y" since it is possible that these more general estimates might be of use in obtaining improvements of the main results of the present paper.

Proposition 3.5. *Let $x \in \mathbb{R}_{>0}, y \in \mathbb{R}_{\geq 2}$. Then the following inequality holds:*

$$\frac{(\log x)^{\pi(y)}}{\pi(y)!\cdot\left(\prod_{i=1}^{\pi(y)} \log p_i\right)} < \Psi(x,y) + 1 \leq \frac{(\log x)^{\pi(y)}}{\pi(y)!\cdot\left(\prod_{i=1}^{\pi(y)} \log p_i\right)} \left(1 + \sum_{i=1}^{\pi(y)} \frac{\log p_i}{\log x}\right)^{\pi(y)}.$$

Proof. Let $j \in \mathbb{Z}_{\geq 0}$. Write $t := \pi(y)$ and

$$
\begin{aligned}
\Lambda &:= \{(n_1, \ldots, n_t) \in \mathbb{Z}^t \mid \sum_{i=1}^{t} n_i \log p_i \leq \log x,\\
&\qquad n_i \geq 0 \text{ for } i = 1, \ldots, t\},\\
\Lambda_j &:= \{(n_1, \ldots, n_{t-1}) \in \mathbb{Z}^{t-1} \mid \sum_{i=1}^{t-1} n_i \log p_i \leq \log x - j \log p_t,\\
&\qquad n_i \geq 0 \text{ for } i = 1, \ldots, t-1\},\\
V &:= \{(r_1, \ldots, r_t) \in \mathbb{R}^t \mid \sum_{i=1}^{t} r_i \log p_i \leq \log x,\\
&\qquad r_i \geq 0 \text{ for } i = 1, \ldots, t\},\\
V_j &:= \{(r_1, \ldots, r_{t-1}, r_t) \in \mathbb{R}^t \mid \sum_{i=1}^{t-1} r_i \log p_i \leq \log x - j \log p_t,\\
&\qquad r_i \geq 0 \text{ for } i = 1, \ldots, t-1,\\
&\qquad j \leq r_t < j+1\},\\
\bar{V} &:= \{(r_1, \ldots, r_t) \in \mathbb{R}^t \mid \sum_{i=1}^{t} r_i \log p_i \leq \log x + \sum_{i=1}^{t} \log p_i,\\
&\qquad r_i \geq 0 \text{ for } i = 1, \ldots, t\}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\sharp\Lambda &= \Psi(x, y) + 1,\\
\mu(V) &= \tfrac{(\log x)^t}{t! \cdot \left(\prod_{i=1}^{t} \log p_i\right)},\\
\mu(V_j) &= \tfrac{(\log x - j \log p_t)^{t-1}}{(t-1)! \cdot \left(\prod_{i=1}^{t-1} \log p_i\right)},\\
\mu(\bar{V}) &= \tfrac{\left(\log x + \sum_{i=1}^{t} \log p_i\right)^t}{t! \cdot \left(\prod_{i=1}^{t} \log p_i\right)} = \tfrac{(\log x)^t}{t! \cdot \left(\prod_{i=1}^{t} \log p_i\right)} \left(1 + \sum_{i=1}^{t} \tfrac{\log p_i}{\log x}\right)^t,
\end{aligned}
$$

where we write μ for a Lebesgue measure.

In the following, we compare $\sharp\Lambda$ with $\mu(V)$ and $\mu(\bar{V})$.

First, let us prove that

$$\mu(V) < \sharp\Lambda.$$

We shall use induction on t.

The case where $t = 1$ is clear, since, in this case, $\sharp\Lambda$ is the smallest integer which is larger than $\frac{\log x}{\log p_1}$.

Next, we consider the case where $t \geq 2$. It follows from the induction hypothesis that

$$\mu(V_j) < \sharp\Lambda_j.$$

Thus, we obtain the inequality

$$\mu(V) \leq \sum_{j=0}^{\infty} \mu(V_j) < \sum_{j=0}^{\infty} \sharp\Lambda_j = \sharp\Lambda.$$

Next, let us prove that

$$\sharp\Lambda \le \mu(\bar{V}).$$

Write

$$I_{(n_1,\dots,n_t)} := \prod_{i=1}^{t} [n_i, n_i+1) \subseteq \mathbb{R}^t.$$

Since $\mu(I_{(n_1,\dots,n_t)}) = 1$, it clearly follows that

$$\sharp\Lambda = \mu\left(\bigcup_{(n_1,\dots,n_t)\in\Lambda} I_{(n_1,\dots,n_t)}\right) \le \mu(\bar{V}).$$

This completes the proof. □

By restricting the size of y and applying Corollary 3.3, we obtain the following two corollaries. The first one was obtained by V. Ennola [E] (cf. [N], p.25). Readers may skip it because it is not used in the present paper.

Corollary 3.6. *Let $x, y \in \mathbb{R}_{>0}$ be such that $2 \le y \le (\log x)^{\frac{1}{2}}$. Then the following estimate holds:*

$$\Psi(x,y) + 1 = \frac{(\log x)^{\pi(y)}}{\pi(y)! \cdot \left(\prod_{i=1}^{\pi(y)} \log p_i\right)} \left(1 + O\left(\frac{y^2}{\log x \log y}\right)\right).$$

Proof. We apply the inequalities of Proposition 3.5. Thus, it suffices to estimate the expression

$$\left(1 + \sum_{i=1}^{\pi(y)} \frac{\log p_i}{\log x}\right)^{\pi(y)} = \exp\left(\pi(y) \log\left(1 + \sum_{i=1}^{\pi(y)} \frac{\log p_i}{\log x}\right)\right).$$

Since $\log(1+z) \le z$ for $z \in \mathbb{R}_{>0}$ (recall that the function $z \mapsto \log(1+z)$ is concave),

$$\exp\left(\pi(y) \log\left(1 + \sum_{i=1}^{\pi(y)} \frac{\log p_i}{\log x}\right)\right) \le \exp\left(\pi(y) \sum_{i=1}^{\pi(y)} \frac{\log p_i}{\log x}\right),$$

and it follows immediately from Corollary 3.4 that

$$\exp\left(\pi(y) \sum_{i=1}^{\pi(y)} \frac{\log p_i}{\log x}\right) = \exp\left(\frac{\pi(y)\theta(y)}{\log x}\right) = \exp\left(O\left(\frac{y\pi(y)}{\log x}\right)\right).$$

Note that, for $M \in \mathbb{R}_{>0}$, there exists a $C \in \mathbb{R}_{>0}$ such that $\exp(z) \le 1 + Cz$ for $0 \le z \le M$ (recall that the function $z \mapsto \exp(z)$ is convex). Since the assumption that $2 \le y \le (\log x)^{\frac{1}{2}}$ implies that

$$\exp\left(\frac{y\pi(y)}{\log x}\right) \le \frac{y^2}{\log x} \le 1,$$

we obtain the estimate

$$\exp\left(O\left(\frac{y\pi(y)}{\log x}\right)\right) = 1 + O\left(\frac{y\pi(y)}{\log x}\right).$$

Finally, it follows from Corollary 3.3 that

$$1 + O\left(\frac{y\pi(y)}{\log x}\right) = 1 + O\left(\frac{y^2}{\log x \log y}\right).$$

This completes the proof. □

The second corollary is an exponential version of Proposition 3.5, obtained by V. Ennola [E] (cf. [N], p. 25).

Corollary 3.7. *Let $x, y, \gamma \in \mathbb{R}_{>0}$ be such that $2 \leq y \leq (\log x)^\gamma$ and $\gamma < 1$. Then the following estimate holds:*

$$\Psi(x, y) = \exp\left(\pi(y)\log\log x - y + O_\gamma\left(\tfrac{y}{(\log y)^2}\right)\right).$$

Proof. We apply the inequalities of Proposition 3.5. Since

$$\frac{(\log x)^{\pi(y)}}{\pi(y)!\cdot\left(\prod_{i=1}^{\pi(y)}\log p_i\right)} = \exp\left(\pi(y)\log\log x - \sum_{i=1}^{\pi(y)}\log i - \sum_{i=1}^{\pi(y)}\log\log p_i\right),$$

it suffices to estimate the expressions

$$\sum_{i=1}^{\pi(y)}\log i,\ \sum_{i=1}^{\pi(y)}\log\log p_i,\ \left(1 + \sum_{i=1}^{\pi(y)}\tfrac{\log p_i}{\log x}\right)^{\pi(y)}.$$

First, let us estimate the expression $\sum_{i=1}^{\pi(y)}\log i$. By applying Stirling's formula, we obtain the estimate

$$\sum_{i=1}^{\pi(y)}\log i = \left(\pi(y) + \frac{1}{2}\right)\log\pi(y) - \pi(y) + O(1).$$

Moreover, by applying Corollary 3.3, together with the estimate $\log(1+\frac{M}{\log y}) \leq \frac{M}{\log y}$ for $M \in \mathbb{R}_{>0}$, we obtain the estimates

$$\log\pi(y) = \log y - \log\log y + O\left(\tfrac{1}{\log y}\right),$$
$$\pi(y) + \frac{1}{2} = \left(\tfrac{y}{\log y} + \tfrac{y}{(\log y)^2} + O\left(\tfrac{y}{(\log y)^3}\right)\right).$$

Then it follows from the above two estimates that

$$\left(\pi(y) + \frac{1}{2}\right)\log\pi(y) = y + \tfrac{y}{\log y} - \tfrac{y\log\log y}{\log y} - \tfrac{y\log\log y}{(\log y)^2} + O\left(\tfrac{y}{(\log y)^2}\right).$$

Thus, we obtain the following estimate

$$\sum_{i=1}^{\pi(y)}\log i = y - \tfrac{y\log\log y}{\log y} - \tfrac{y\log\log y}{(\log y)^2} + O\left(\tfrac{y}{(\log y)^2}\right).$$

Next, let us estimate $\sum_{i=1}^{\pi(y)}\log\log p_i$. By computing Lebesgue-Stieltjes integrals, it follows that

$$\sum_{i=1}^{\pi(y)}\log\log p_i = \int_{2-0}^{y}\log\log t\ d\pi(t) = \pi(y)\log\log y - \int_{2-0}^{y}\tfrac{\pi(t)}{t\log t}dt.$$

By applying Corollary 3.3 and Lemma 3.2, we obtain the following two estimates

$$\pi(y)\log\log y = \tfrac{y\log\log y}{\log y} + \tfrac{y\log\log y}{(\log y)^2} + O\left(\tfrac{y}{(\log y)^2}\right),$$
$$\int_{2-0}^{y}\tfrac{\pi(t)}{t\log t}dt = \int_{2}^{y}O\left(\tfrac{1}{(\log t)^2}\right)dt = O\left(\tfrac{y}{(\log y)^2}\right).$$

Thus, it follows that

$$\sum_{i=1}^{\pi(y)} \log\log p_i = \frac{y\log\log y}{\log y} + \frac{y\log\log y}{(\log y)^2} + O\left(\frac{y}{(\log y)^2}\right).$$

By combining the above estimates, it immediately follows that

$$\frac{(\log x)^{\pi(y)}}{\pi(y)!\cdot\left(\prod_{i=1}^{\pi(y)}\log p_i\right)} = \exp\left(\pi(y)\log\log x - y + O\left(\frac{y}{(\log y)^2}\right)\right).$$

Finally, let us estimate $\left(1+\sum_{i=1}^{\pi(y)}\frac{\log p_i}{\log x}\right)^{\pi(y)}$. By a similar calculation to the calculation applied in the proof of Corollary 3.6, we obtain the estimate

$$\left(1+\sum_{i=1}^{\pi(y)}\frac{\log p_i}{\log x}\right)^{\pi(y)} = \exp\left(O\left(\frac{y\pi(y)}{\log x}\right)\right).$$

Since

$$\frac{\pi(y)}{\log x} = O\left(\frac{y}{y^{\frac{1}{\gamma}}}\right) = O_\gamma\left(\frac{y}{(\log y)^2}\right),$$

it follows immediately that

$$\left(1+\sum_{i=1}^{\pi(y)}\frac{\log p_i}{\log x}\right)^{\pi(y)} = \exp\left(O_\gamma\left(\frac{y}{(\log y)^2}\right)\right).$$

This completes the proof. □

Finally, we give estimates for various versions of Ψ.

Theorem 3.8. *Let $x, y, \gamma \in \mathbb{R}_{>0}$ such that $2 \le y = (\log x)^\gamma$ and $\gamma < 1$. Then the following estimate holds:*

$$\begin{aligned}\Psi(x,y) &= \exp\left(\left(\frac{1}{\gamma}-1\right)y + \frac{1}{\gamma}\frac{y}{\log y} + O_\gamma\left(\frac{y}{(\log y)^2}\right)\right)\\ &= \exp\left(\left(\frac{1}{\gamma}-1\right)(\log x)^\gamma + \frac{1}{\gamma^2}\frac{(\log x)^\gamma}{\log\log x} + O_\gamma\left(\frac{(\log x)^\gamma}{(\log\log x)^2}\right)\right).\end{aligned}$$

Proof. By applying Corollary 3.3 and Corollary 3.7, we obtain the following estimate

$$\begin{aligned}\Psi(x,y) &= \exp\left(\frac{1}{\gamma}\pi(y)\log y - y + O_\gamma\left(\frac{y}{(\log y)^2}\right)\right)\\ &= \exp\left(\frac{1}{\gamma}\left(y + \frac{y}{\log y} + O\left(\frac{y}{(\log y)^2}\right)\right) - y + O_\gamma\left(\frac{y}{(\log y)^2}\right)\right)\\ &= \exp\left(\left(\frac{1}{\gamma}-1\right)y + \frac{1}{\gamma}\frac{y}{\log y} + O_\gamma\left(\frac{y}{(\log y)^2}\right)\right)\\ &= \exp\left(\left(\frac{1}{\gamma}-1\right)(\log x)^\gamma + \frac{1}{\gamma^2}\frac{(\log x)^\gamma}{\log\log x} + O_\gamma\left(\frac{(\log x)^\gamma}{(\log\log x)^2}\right)\right).\end{aligned}$$

This completes the proof. □

Theorem 3.9. *Let $x, y, \gamma \in \mathbb{R}_{>0}$, $u \in \mathbb{Z}_{\ge 1}$, and $q_1, ..., q_u \in \mathfrak{Primes}$ such that, for $i = 1, \dots, u$, $2 \le q_i \le y = (\log x)^\gamma$ and $\gamma < 1$. Write $D := \prod_{i=1}^u q_i$. Then the* *following estimate holds:*

$$\begin{aligned}\Psi(x,y;D) &= \exp\left(\left(\tfrac{1}{\gamma}-1\right)y + \tfrac{1}{\gamma}\tfrac{y}{\log y} + O_{\gamma,u}\left(\tfrac{y}{(\log y)^2}\right)\right)\\ &= \exp\left(\left(\tfrac{1}{\gamma}-1\right)(\log x)^{\gamma} + \tfrac{1}{\gamma^2}\tfrac{(\log x)^{\gamma}}{\log\log x} + O_{\gamma,u}\left(\tfrac{(\log x)^{\gamma}}{(\log\log x)^2}\right)\right).\end{aligned}$$

Proof. First, we introduce some notation. Let $z \in \mathbb{R}$. We denote the largest integer $\leq z$ by $\lfloor z \rfloor$. Let $w_1 \in \mathbb{R}_{>0} \setminus \{1\}$, $w_2 \in \mathbb{R}_{>0}$. Then we shall write $\log_{w_1} w_2 := \frac{\log w_2}{\log w_1}$. Note that $w_1^{\log_{w_1} w_2} = w_2$.

It follows immediately from the definitions that $\Psi(x,y;D) \leq \Psi(x,y)$. Next, by classifying the "x''s" that occur in the definition of "$\Psi(x,y)$" by the extent of their divisibility by the q_i's, we obtain the following estimate:

$$\begin{aligned}\Psi(x,y) &= \sum_{j_1=0}^{\infty}\cdots\sum_{j_u=0}^{\infty}\ \Psi\left(\left(\prod_{i=1}^{u} q_i^{-j_i}\right)x, y; D\right)\\ &= \sum_{j_1=0}^{\lfloor \log_{q_1} x\rfloor}\cdots\sum_{j_u=0}^{\lfloor \log_{q_u} x\rfloor}\ \Psi\left(\left(\prod_{i=1}^{u} q_i^{-j_i}\right)x, y; D\right)\\ &\leq \left(\prod_{i=1}^{u}\left(\lfloor \log_{q_i} x\rfloor + 1\right)\right)\Psi(x,y;D).\end{aligned}$$

On the other hand, since for $p \in \mathfrak{Primes}$, $\log p^2 \geq 1$, and $\log x \geq 1$,

$$\begin{aligned}\prod_{i=1}^{u}\left(\lfloor \log_{q_i} x\rfloor + 1\right) &= \exp\left(\sum_{i=1}^{u}\log\left(\lfloor \log_{q_i} x\rfloor + 1\right)\right)\\ &\leq \exp\left(\sum_{i=1}^{u}\log(2\log x + 1)\right)\\ &\leq \exp\left(u(\log(3\log x))\right)\\ &= \exp\left(O_{\gamma,u}\left(\tfrac{(\log x)^{\gamma}}{(\log\log x)^2}\right)\right).\end{aligned}$$

Thus, it follows from Theorem 3.8 that

$$\begin{aligned}\Psi(x,y;D) &= \exp\left(\left(\tfrac{1}{\gamma}-1\right)(\log x)^{\gamma} + \tfrac{1}{\gamma^2}\tfrac{(\log x)^{\gamma}}{\log\log x} + O_{\gamma,u}\left(\tfrac{(\log x)^{\gamma}}{(\log\log x)^2}\right)\right)\\ &= \exp\left(\left(\tfrac{1}{\gamma}-1\right)y + \tfrac{1}{\gamma}\tfrac{y}{\log y} + O_{\gamma,u}\left(\tfrac{y}{(\log y)^2}\right)\right).\end{aligned}$$

This completes the proof. □

4. **Proof of Theorem 2.1**

First, for ease of reference, we review the statement of Theorem 2.1:

> *Let $r \in \mathbb{Q}$; $\varepsilon, N_0, \gamma \in \mathbb{R}_{>0}$ such that $\gamma < \frac{1}{2}$; $\Sigma \subseteq \mathbb{V}$ a finite subset which includes ∞; and $K_{r,\varepsilon,\Sigma}$ an (r,ε,Σ)-compactly bounded subset (cf. Definition 1.6). Then there exists an abc-triple (a,b,c) such that*
>
> $$N_{(a,b,c)} > N_0,$$

$$|abc| > N^3_{(a,b,c)} \exp\left((\log\log N_{(a,b,c)})^{\frac{1}{2}-\gamma}\right),$$

$$\lambda_{(a,b,c)} \in K_{r,\varepsilon,\Sigma}.$$

Before beginning the proof, we recall from §1 that, for $r \in \mathbb{Q}$, $\varepsilon \in \mathbb{R}_{>0}$, $\infty \in \Sigma \subseteq \mathbb{V}$ such that Σ is a finite subset,

$$K_{r,\varepsilon,\Sigma} := \{r' \in \mathbb{Q} \mid \|r' - r\|_v < \varepsilon, \forall v \in \Sigma\}.$$

It follows immediately from the definition of "$K_{r,\varepsilon,\Sigma}$" that, given a finite subset $\Xi \subseteq \mathbb{Q}$, we may assume without loss of generality, in the statement of Theorem 2.1, that $r \notin \Xi$ and $\varepsilon < 1$. In particular, by taking Ξ to be $\{0,1\}$ we may assume in the following that $r \neq 0, 1$. Next, let us recall that $\lambda_{(a,b,c)} := -\frac{b}{a}$. Since, for every *abc*-triple (a,b,c), $\lambda_{(a,c,b)} = 1 - \lambda_{(a,b,c)}$, and $\lambda_{(b,a,c)} = \frac{1}{\lambda_{(a,b,c)}}$, we may assume without loss of generality, in the statement of Theorem 2.1, that $r > 1$. Finally, in a similar vein, it follows immediately from the definition of "$K_{r,\varepsilon,\Sigma}$" that we may assume without loss of generality, in the statement of Theorem 2.1, that $r - \varepsilon > 1$.

Next, we introduce notation as follows:

- Write
$$\Sigma_f := \Sigma \setminus \{\infty\}.$$

- Let $\delta \in \mathbb{R}_{>0}$ be such that
$$\delta < 12.$$
Then observe that there exists a $\delta' \in \mathbb{R}_{>0}$ such that
$$\delta' < 12, \quad \frac{12-\delta}{(1+3\delta)^{\frac{1}{2}}} > 12 - \delta'.$$

- Write
$$D := \prod_{v \in \Sigma_f} p_v$$
(so $D = 1$ if $\Sigma_f = \emptyset$).

- We define $q \in \mathfrak{Primes}$ to be the smallest odd prime number such that
$$q > N_0$$
and, for $v \in \Sigma_f$,
$$q \neq p_v, \ \|r\|_w = 1,$$
where we write
$$w \in \mathbb{V}$$
for the q-adic valuation on $\mathbb{Q}$.

- We define
$$\varepsilon' := \frac{\varepsilon}{\max\{\|r\|_v\}_{v \in \Sigma}} \leq \frac{\varepsilon}{r} < \varepsilon.$$

- We define $J \in \mathbb{Z}_{\geq 1}$ to be the smallest positive integer such that, for $v \in \Sigma_f$,

$$\frac{1}{p_v^J} \leq \varepsilon',$$

$g \in \mathbb{Z}_{\geq 1}$ to be the smallest positive integer such that

$$\exp\left(\frac{1}{g}\right) \leq 1 + \varepsilon'.$$

- In the following discussion, we shall construct an element

$$x_0 \in \mathbb{R}_{>3}$$

which depends only on r, ε, Σ, N_0, and δ. Note that D, q, ε', J, and g depend only on r, ε, Σ, and N_0. Let

$$x \in \mathbb{R}_{>x_0}.$$

Write

$$y := (\log x)^{\frac{1}{2}}.$$

We define $G \in \mathbb{Z}_{\geq 1}$ to be the smallest positive integer such that

$$G > g \log x.$$

Thus, for a suitable choice of x_0, it follows from Theorem 3.9 (where we take γ to be $\frac{1}{2}$) that

$$\Psi(x, y; D \cdot q) = \exp\left((\log x)^{\frac{1}{2}} + 4\frac{(\log x)^{\frac{1}{2}}}{\log \log x} + O_{\sharp\Sigma}\left(\frac{(\log x)^{\frac{1}{2}}}{(\log \log x)^2}\right)\right). \qquad (\dagger_1)$$

- Observe that there exists a unique $I \in \mathbb{Z}$ such that

$$\frac{1}{q}\Psi(x, y; D \cdot q) \leq GD^J q^I < \Psi(x, y; D \cdot q). \qquad (\dagger_2)$$

It follows immediately from the estimate ($\dagger_1$) that, for a suitable choice of x_0, we may suppose that $I \geq 1$.

Lemma 4.1. *For a suitable choice of x_0, there exists a pair of positive integers (a_1, b_1) such that*

(1) $$\mathrm{LPN}(a_1) \leq y,\ \mathrm{LPN}(b_1) \leq y,$$

(2) $$(a_1, b_1) = 1,\ (a_1, D \cdot q) = 1,\ (b_1, D \cdot q) = 1,$$

(3) $$1 \leq a_1 \leq x,\ 1 \leq b_1 \leq x,$$

(4) $$\|\tfrac{b_1}{a_1} - 1\|_v \leq \varepsilon' \text{ for } v \in \Sigma_f,$$

(5) $$1 < \tfrac{b_1}{a_1} < 1 + \varepsilon',$$

(6) $$\|\tfrac{b_1}{a_1} - 1\|_w \leq \tfrac{1}{q^I}.$$

Proof. First, let us recall the estimate ($\dagger_2$)

$$GD^J q^I < \Psi(x, y; D \cdot q).$$

Thus, by considering the residue classes modulo $D^J q^I$ of the set of integers that appears in the definition of $\Psi(x, y; D \cdot q)$, we conclude from the Box Principle that there exists a sequence of $G+1$ integers $2 \leq s_0 < \cdots < s_G \leq x$ such that

$$\mathrm{LPN}(s_i) \leq y \text{ for } i = 0, \dots, G,$$
$$(s_i, D \cdot q) = 1 \text{ for } i = 0, \dots, G,$$
$$s_i \equiv s_j \bmod D^J q^I \text{ for } i, j = 0, \dots, G.$$

Next, let us suppose that $s_{i+1} > x^{\frac{1}{g \log x}} \cdot s_i$ for $i = 0, \dots, G-1$. Since $G > g \log x$, it follows immediately that

$$x \geq s_G > x^{\frac{1}{g \log x}} \cdot s_{G-1} > \cdots > x^{\frac{G}{g \log x}} \cdot s_0 > x s_0 \geq x$$

— a contradiction. Thus, there exists an $i_0 \in \mathbb{Z}$ such that

$$0 \leq i_0 \leq G - 1,$$
$$s_{i_0} < s_{i_0+1} \leq x^{\frac{1}{g \log x}} s_{i_0}.$$

Since $x^{\frac{1}{g \log x}} = \exp\left(\frac{1}{g}\right) < 1 + \varepsilon'$, it follows that

$$s_{i_0} < s_{i_0+1} < (1 + \varepsilon') s_{i_0}.$$

We define $a_1, b_1 \in \mathbb{Z}_{\geq 1}$ as follows:

$$(a_1, b_1) = 1,$$
$$\tfrac{b_1}{a_1} := \tfrac{s_{i_0+1}}{s_{i_0}}.$$

Then it follows immediately from the definition of (a_1, b_1) that (a_1, b_1) satisfies conditions (1), (2), (3), (4), (5), and (6) of Lemma 4.1. This completes the proof. $\square$

Lemma 4.2. *For a suitable choice of x_0, there exists a pair of positive integers (a_2, b_2) such that*

$$\mathrm{LPN}(a_2) \leq y,\ \mathrm{LPN}(b_2) \leq y, \tag{1}$$
$$(a_2, b_2) = 1,\ (a_2, q) = 1,\ (b_2, q) = 1, \tag{2}$$
$$1 \leq a_2 \leq x,\ 1 \leq b_2 \leq \exp\left(\exp\left(3(\log x)^{\frac{1}{2}}\right)\right), \tag{3}$$
$$\|\tfrac{b_2}{a_2} - r\|_v \leq \varepsilon \text{ for } v \in \Sigma_f, \tag{4}$$
$$\|\tfrac{b_2}{a_2} - 1\|_w \leq \tfrac{1}{q^I}. \tag{5}$$

Proof. First, since q is an odd prime number, there exists an $h_0 \in \mathbb{Z}$ such that $(\mathbb{Z}/q^2\mathbb{Z})^\times$ is generated as a group by the image of h_0 in $(\mathbb{Z}/q^2\mathbb{Z})^\times$. Note that $(\mathbb{Z}/q^I\mathbb{Z})^\times$ is also generated as a group by the image of h_0 in $(\mathbb{Z}/q^I\mathbb{Z})^\times$. Thus, it follows from the Chinese Remainder Theorem that there exists an $h \in \mathbb{Z}_{\geq 1}$ such that, for $v \in \Sigma_f$,

$$h \equiv 1 \bmod D^J,\ h \equiv h_0 \bmod q^2.$$

Next, since $\|r\|_w = 1$, there exists a $u \in \mathbb{Z}$ such that

$$\|u - \tfrac{1}{r}\|_w \leq \tfrac{1}{q^I} < 1.$$

Thus, since $(u,q)=1$, and the image of h in $(\mathbb{Z}/q^I\mathbb{Z})^\times$ generates $(\mathbb{Z}/q^I\mathbb{Z})^\times$, it follows that there exists a positive integer $n \leq q^I$ such that

$$h^n \equiv 1 \bmod D^J,\ h^n \equiv u \bmod q^I.$$

Let us estimate h^n. First, for a suitable choice of x_0, we have

$$h < y.$$

Next, it follows from the definition of G and the inequality $(\dagger_2)$ that

$$\begin{aligned} h^n < y^{q^I} &= \exp\left(q^I \log y\right) < \exp\left(\tfrac{\log\log x}{2D^J G}\Psi(x,y;D\cdot q)\right) \\ &< \exp\left(\tfrac{\log\log x}{2D^J g\log x}\Psi(x,y;D\cdot q)\right). \end{aligned}$$

Finally, it follows from the estimate $(\dagger_1)$ that, for a suitable choice of x_0,

$$\exp\left(\tfrac{\log\log x}{2D^J g\log x}\Psi(x,y;D\cdot q)\right) < \exp\left(\exp\left(2(\log x)^{\frac{1}{2}}\right)\right).$$

Thus, it follows that

$$h^n < \exp\left(\exp\left(2(\log x)^{\frac{1}{2}}\right)\right).$$

We define the pair of positive integers (a_2,b_2) as follows:

$$(a_2,b_2)=1,$$

$$\tfrac{b_2}{a_2} := rh^n.$$

Since, for $v \in \Sigma_f$,

$$\|rh^n - r\|_v = \|r\|_v \cdot \|h^n - 1\|_v \leq \|r\|_v \cdot \varepsilon' \leq \varepsilon,$$

$$\|rh^n - 1\|_w = \|r\|_w \cdot \|h^n - u + u - \tfrac{1}{r}\|_w \leq \max\{\|h^n-u\|_w, \|u-\tfrac{1}{r}\|_w\} \leq \tfrac{1}{q^I},$$

it follows immediately from the definition of (a_2,b_2) that (a_2,b_2) satisfies conditions (2), (4), and (5) of Lemma 4.2.

Let $r_a, r_b \in \mathbb{Z}_{\geq 1}$ such that

$$(r_a,r_b)=1,\ r=\tfrac{r_b}{r_a}.$$

For a suitable choice of x_0, it follows that

$$r_a < y < x,\ r_b < y < \exp\left(\exp\left((\log x)^{\frac{1}{2}}\right)\right).$$

Thus, it follows that

$$1 \leq a_2 \leq r_a < x,\ 1 \leq b_2 \leq r_b h^n < \exp\left(\exp\left((1+2)(\log x)^{\frac{1}{2}}\right)\right),$$

and hence that (a_2,b_2) satisfies conditions (1) and (3) of Lemma 4.2. This completes the proof. $\square$

Lemma 4.3. *For a suitable choice of x_0, there exist a pair of positive integers (a_3, b_3) and an $\alpha \in \mathbb{Z}_{\geq 0}$ such that*

$$\mathrm{LPN}(a_3) \leq y,\ \mathrm{LPN}(b_3) \leq y, \tag{1}$$

$$(a_3, b_3) = 1,\ (a_3, q) = 1,\ (b_3, q) = 1, \tag{2}$$

$$1 \leq a_3 \leq x^{\alpha+1},\ 1 \leq b_3 \leq x^{\alpha} \exp\left(\exp\left(3(\log x)^{\frac{1}{2}}\right)\right), \tag{3}$$

$$\|\tfrac{b_3}{a_3} - r\|_v \leq \varepsilon \textit{ for } v \in \Sigma, \tag{4}$$

$$\|\tfrac{b_3}{a_3} - 1\|_w \leq \tfrac{1}{q^I}, \tag{5}$$

$$0 \leq \alpha \leq x^{1+\delta}. \tag{6}$$

Proof. It follows from Lemmas 4.1 and 4.2 that, for a suitable choice of x_0, there exist a pair of positive integers (a_1, b_1) which satisfies the conditions of Lemma 4.1, and a pair of positive integers (a_2, b_2) which satisfies the conditions of Lemma 4.2.

Since

$$1 < \tfrac{b_1}{a_1} < 1 + \varepsilon' \leq 1 + \tfrac{\varepsilon}{r} = \tfrac{r+\varepsilon}{r} \leq \tfrac{r+\varepsilon}{r-\varepsilon},$$

it follows immediately, by considering the elementary geometry of the real line, that there exists an $\alpha' \in \mathbb{Z}$ such that

$$\|\tfrac{b_2}{a_2}\left(\tfrac{b_1}{a_1}\right)^{\alpha'} - r\|_\infty \leq \varepsilon.$$

Next, let us prove that, for $v \in \Sigma_f$,

$$\|\tfrac{b_2}{a_2}\left(\tfrac{b_1}{a_1}\right)^{\alpha'} - r\|_v \leq \varepsilon, \quad \|\tfrac{b_2}{a_2}\left(\tfrac{b_1}{a_1}\right)^{\alpha'} - 1\|_w \leq \tfrac{1}{q^I}.$$

Since $v \in \Sigma_f$ is a non-archimedean valuation on $\mathbb{Q}$, and $(a_1, D\cdot q) = 1$, $(b_1, D\cdot q) = 1$, $\|\frac{b_1}{a_1} - 1\|_v \leq \varepsilon'$, $\|\frac{b_1}{a_1} - 1\|_w \leq \frac{1}{q^I}$ (cf. conditions (2), (4), and (6) of Lemma 4.1), it follows that, for $v \in \Sigma_f$,

$$\|\left(\tfrac{b_1}{a_1}\right)^{\alpha'} - 1\|_v \leq \varepsilon', \quad \|\left(\tfrac{b_1}{a_1}\right)^{\alpha'} - 1\|_w \leq \tfrac{1}{q^I}.$$

Thus, since $\varepsilon' = \frac{\varepsilon}{\max\{\|r\|_v\}_{v\in\Sigma}} \leq \frac{\varepsilon}{\|r\|_v}$, and $(a_1, D\cdot q) = 1$, $(b_1, D\cdot q) = 1$, $\|\frac{b_2}{a_2} - r\|_v \leq \varepsilon$, $\|\frac{b_2}{a_2} - 1\|_w \leq \frac{1}{q^I}$ (cf. condition (2) of Lemma 4.1 and conditions (4) and (5) of Lemma 4.2), it follows that, for $v \in \Sigma_f$,

$$\begin{aligned}
\left\|\tfrac{b_2}{a_2}\left(\tfrac{b_1}{a_1}\right)^{\alpha'} - r\right\|_v &\leq \max\left\{\left\|\left(\tfrac{b_2}{a_2} - r\right)\left(\tfrac{b_1}{a_1}\right)^{\alpha'}\right\|_v,\ \left\|\left(\left(\tfrac{b_1}{a_1}\right)^{\alpha'} - 1\right) r\right\|_v\right\} \\
&\leq \max\{\varepsilon,\ \varepsilon'\|r\|_v\} \leq \varepsilon, \\
\left\|\tfrac{b_2}{a_2}\left(\tfrac{b_1}{a_1}\right)^{\alpha'} - 1\right\|_w &\leq \max\left\{\left\|\left(\tfrac{b_2}{a_2} - 1\right)\left(\tfrac{b_1}{a_1}\right)^{\alpha'}\right\|_w,\ \left\|\left(\tfrac{b_1}{a_1}\right)^{\alpha'} - 1\right\|_w\right\} \\
&\leq \tfrac{1}{q^I}.
\end{aligned}$$

We define $a_3, b_3 \in \mathbb{Z}_{\geq 1}, \alpha \in \mathbb{Z}_{\geq 0}$ as follows:

$$(a_3, b_3) = 1,$$

$$\tfrac{b_3}{a_3} = \tfrac{b_2}{a_2}\left(\tfrac{b_1}{a_1}\right)^{\alpha'},$$

$$\alpha := |\alpha'|.$$

It follows immediately from the definitions of (a_3, b_3) and α that (a_3, b_3) and α satisfy conditions (1), (2), (3), (4), and (5) of Lemma 4.3.

Next, let us estimate α. First, since $\frac{1}{x} \leq \frac{b_2}{a_2} \leq \exp\left(\exp\left(3(\log x)^{\frac{1}{2}}\right)\right)$ (cf. condition (3) of Lemma 4.2), and $1 < r - \varepsilon \leq \frac{b_3}{a_3} \leq r + \varepsilon$, it follows that

$$\left(\frac{b_1}{a_1}\right)^{\alpha'} = \left(\frac{b_2}{a_2}\right)^{-1} \frac{b_3}{a_3} \leq x(r+\varepsilon),$$

$$\left(\frac{b_1}{a_1}\right)^{\alpha'} = \left(\frac{b_2}{a_2}\right)^{-1} \frac{b_3}{a_3} > \exp\left(-\exp\left(3(\log x)^{\frac{1}{2}}\right)\right).$$

Next, observe that, for a suitable choice of x_0, we have

$$r + \varepsilon < x.$$

Thus, it follows that

$$\exp\left(-\exp\left(3(\log x)^{\frac{1}{2}}\right)\right) < \left(\frac{b_1}{a_1}\right)^{\alpha'} < x^2,$$

i.e.,

$$-\exp\left(3(\log x)^{\frac{1}{2}}\right) < \alpha' \log\left(\frac{b_1}{a_1}\right) < 2\log x.$$

In particular, it follows immediately from the above estimate that

$$\alpha \log\left(\frac{b_1}{a_1}\right) = |\alpha' \log\left(\frac{b_1}{a_1}\right)| < \max\left\{\exp\left(3(\log x)^{\frac{1}{2}}\right), 2\log x\right\}.$$

Since, for a suitable choice of x_0,

$$2\log x < \exp\left(3(\log x)^{\frac{1}{2}}\right),$$

we thus conclude that, for a suitable choice of x_0,

$$\alpha \log\left(\frac{b_1}{a_1}\right) < \exp\left(3(\log x)^{\frac{1}{2}}\right).$$

Moreover, since $1 + \frac{1}{x} \leq 1 + \frac{1}{a_1} = \frac{a_1+1}{a_1} \leq \frac{b_1}{a_1}$ (cf. conditions (3) and (5) of Lemma 4.1), and $\log 2 \leq \log\left(1 + \frac{1}{x}\right)^x$ for $x \geq 1$, it follows that

$$\frac{\alpha \log 2}{x} \leq \alpha \log\left(1 + \frac{1}{x}\right) \leq \alpha \log\left(\frac{b_1}{a_1}\right) < \exp\left(3(\log x)^{\frac{1}{2}}\right).$$

Thus, it follows that, for a suitable choice of x_0,

$$\alpha < x^{1+\delta},$$

i.e., α satisfies condition (6) of Lemma 4.3. This completes the proof. $\square$

Lemma 4.4. *There exists an abc-triple* (a, b, c) *such that*

$$N_{(a,b,c)} > N_0, \tag{1}$$

$$|abc| > N^3_{(a,b,c)} \exp\left((12 - \delta') \frac{(\log\log N_{(a,b,c)})^{\frac{1}{2}}}{\log\log\log N_{(a,b,c)}}\right), \tag{2}$$

$$\lambda_{(a,b,c)} \in K_{r,\varepsilon,\Sigma}. \tag{3}$$

Proof. It follows from Lemma 4.3 that, for a suitable choice of x_0, there exist a pair of positive integers (a_3, b_3) and an $\alpha \in \mathbb{Z}_{\geq 0}$ which satisfy the conditions of Lemma 4.3.

Let

$$a := a_3,\ b := -b_3,\ c := -a_3 + b_3.$$

Since $\frac{b_3}{a_3} \geq r - \varepsilon > 1$ (cf. condition (4) of Lemma 4.3), $c \neq 0$. Thus, it follows from condition (2) of Lemma 4.3 that (a, b, c) is an *abc*-triple. Next, observe that, since $I \geq 1$, it follows from conditions (2) and (5) of Lemma 4.3 that $q \mid c$. Since $q \mid c$ and $N_0 < q \in \mathfrak{Primes}$, it follows that (a, b, c) satisfies condition (1) of Lemma 4.4, i.e., $N_{(a,b,c)} > N_0$. Finally, since $\lambda_{(a,b,c)} = -\frac{b}{a}$, it follows from condition (4) of Lemma 4.3 that (a, b, c) satisfies condition (3) of Lemma 4.4. Thus, it suffices to show that (a, b, c) satisfies condition (2) of Lemma 4.4.

First, since $\mathrm{LPN}(a_3) \leq y$, $\mathrm{LPN}(b_3) \leq y$ (cf. condition (1) of Lemma 4.3), it follows that

$$\prod_{\substack{p \in \mathfrak{Primes}, \\ p \mid ab}} p \leq \prod_{\substack{p \in \mathfrak{Primes}, \\ p \leq y}} p = \exp(\theta(y)).$$

Next, since $(a_3, q) = 1$ and $\|\frac{b_3}{a_3} - 1\|_w \leq \frac{1}{q^I}$ (cf. the conditions (2) and (5) of Lemma 4.3), and $I \geq 1$, it follows that

$$q^I \mid c.$$

Thus, it follows from the definition of $N_{(a,b,c)}$ that

$$N_{(a,b,c)} = \left(\prod_{\substack{p \in \mathfrak{Primes}, \\ p \mid ab}} p \right) \left(\prod_{\substack{p \in \mathfrak{Primes}, \\ p \mid c}} p \right) \leq \exp(\theta(y)) \cdot \tfrac{|c|}{q^{I-1}}.$$

Next, since the positive integers a_3 and b_3 satisfy the inequalities $\frac{b_3}{a_3} \geq r - \varepsilon > 1$ and $\frac{b_3}{a_3} \leq r + \varepsilon$ (cf. condition (4) of Lemma 4.3), it follows immediately that

$$|b| \geq |c|, \quad |a| \geq \tfrac{1}{r+\varepsilon}|b|.$$

Thus, it follows that

$$|abc| \geq \tfrac{1}{r+\varepsilon}|c|^3 \geq \tfrac{1}{(r+\varepsilon)q^3} \left(N_{(a,b,c)} \exp(-\theta(y)) q^I \right)^3 = C_1 \left(N_{(a,b,c)} \exp(-\theta(y)) q^I \right)^3,$$

where we write $C_1 := \frac{1}{(r+\varepsilon)q^3}$. Next, since $\log x > \log x_0 > \log 3 > 1$, it follows from the inequality $(\dagger_2)$ that

$$q^I \geq \tfrac{\Psi(x,y;D\cdot q)}{GD^J q} \geq \tfrac{\log x}{(g \log x + 1)D^J q} \tfrac{\Psi(x,y;D\cdot q)}{\log x} \geq \tfrac{1}{(g+1)D^J q} \tfrac{\Psi(x,y;D\cdot q)}{\log x} = C_2 \tfrac{\Psi(x,y;D\cdot q)}{\log x},$$

where we write $C_2 := \frac{1}{(g+1)D^J q}$. Thus, it follows that

$$|abc| \geq C_1 C_2^3 \left(\tfrac{N_{(a,b,c)} \exp(-\theta(y)) \Psi(x,y;D\cdot q)}{\log x} \right)^3 = C_3 \left(\tfrac{N_{(a,b,c)} \exp(-\theta(y)) \Psi(x,y;D\cdot q)}{\log x} \right)^3,$$

where we write $C_3 := C_1 C_2^3$. Note that C_3 depends only on r, ε, Σ, and N_0. Thus, it follows from Corollary 3.4 and the estimate ($\dagger_1$) that

$$\begin{aligned}
&\tfrac{\exp(-\theta(y))\Psi(x,y;D\cdot q)}{\log x}\\
= \; &\exp\left(-(\log x)^{\frac{1}{2}} + O\left(\tfrac{(\log x)^{\frac{1}{2}}}{(\log\log x)^2}\right)\right) \times\\
&\exp\left((\log x)^{\frac{1}{2}} + 4\tfrac{(\log x)^{\frac{1}{2}}}{\log\log x} + O_{\sharp\Sigma}\left(\tfrac{(\log x)^{\frac{1}{2}}}{(\log\log x)^2}\right)\right) \times\\
&\exp(-\log\log x)\\
= \; &\exp\left(4\tfrac{(\log x)^{\frac{1}{2}}}{\log\log x} + O_{\sharp\Sigma}\left(\tfrac{(\log x)^{\frac{1}{2}}}{(\log\log x)^2}\right)\right),
\end{aligned}$$

and hence, for a suitable choice of x_0, that

$$|abc| > N_{(a,b,c)}^3 \exp\left((12-\delta)\tfrac{(\log x)^{\frac{1}{2}}}{\log\log x}\right).$$

Finally, let us estimate $N_{(a,b,c)}$. First, it follows from the estimate of $N_{(a,b,c)}$ obtained above, together with the definition of c and the inequality $\frac{b_3}{a_3} > 1$, that

$$N_{(a,b,c)} \leq \exp(\theta(y)) \cdot |c| \leq \exp(\theta(y)) \cdot |b|.$$

Next, for a suitable choice of x_0, it follows from Corollary 3.4 and conditions (3) and (6) of Lemma 4.3 that

$$\exp(\theta(y)) \leq \exp(2(\log x)^{\frac{1}{2}}),$$

$$|b| = b_3 \leq x^{x^{1+\delta}} \exp\left(\exp\left(3(\log x)^{\frac{1}{2}}\right)\right).$$

Next, for a suitable choice of x_0, it follows from an elementary calculation that

$$\exp\left(2(\log x)^{\frac{1}{2}} + \exp\left(3(\log x)^{\frac{1}{2}}\right)\right) \leq x^{x^{\delta}}.$$

Thus, we conclude that, for a suitable choice of x_0,

$$N_{(a,b,c)} \leq x^{x^{1+2\delta}},$$

i.e.,

$$\log\log N_{(a,b,c)} \leq (1+2\delta)\log x + \log\log x.$$

In particular, for a suitable choice of x_0, it holds that

$$\log\log N_{(a,b,c)} \leq (1+3\delta)\log x.$$

Next, observe that we may assume without loss of generality that

$$\log\log N_0 > \exp(2).$$

Since the function

$$z \mapsto \tfrac{z^{\frac{1}{2}}}{\log z} \text{ for } z \in \mathbb{R}_{>\exp(2)}$$

is strictly monotone increasing, $\frac{12-\delta}{(1+3\delta)^{\frac{1}{2}}} > 12-\delta'$, and $N_{(a,b,c)} > N_0$, it follows that, for a suitable choice of x_0,

$$\begin{aligned}
&\exp\left((12-\delta)\tfrac{(\log x)^{\frac{1}{2}}}{\log\log x}\right) \\
= \;& \exp\left(\tfrac{12-\delta}{(1+3\delta)^{\frac{1}{2}}}\cdot\tfrac{\log\log x+\log(1+3\delta)}{\log\log x}\cdot\tfrac{((1+3\delta)\log x)^{\frac{1}{2}}}{\log((1+3\delta)\log x)}\right) \\
> \;& \exp\left((12-\delta')\tfrac{((1+3\delta)\log x)^{\frac{1}{2}}}{\log((1+3\delta)\log x)}\right) \\
\geq \;& \exp\left((12-\delta')\tfrac{(\log\log N_{(a,b,c)})^{\frac{1}{2}}}{\log\log\log N_{(a,b,c)}}\right).
\end{aligned}$$

Thus, it follows that (a,b,c) satisfies condition (2) of Lemma 4.4, i.e.,

$$|abc| > N^3_{(a,b,c)}\exp\left((12-\delta')\tfrac{(\log\log N_{(a,b,c)})^{\frac{1}{2}}}{\log\log\log N_{(a,b,c)}}\right).$$

This completes the proof. □

Proof of Theorem 2.1. Observe that there exists an $M \in \mathbb{R}_{>0}$ such that, for $z \in \mathbb{R}_{>M}$,

$$\tfrac{\log z}{12-\delta'} < z^{\gamma}.$$

Now we apply Lemma 4.4. Observe that we may assume without loss of generality that

$$\log\log N_0 > M.$$

Since $N_{(a,b,c)} > N_0$, it follows that

$$\tfrac{\log\log\log N_{(a,b,c)}}{12-\delta'} < (\log\log N_{(a,b,c)})^{\gamma}.$$

Thus, we conclude that

$$\begin{aligned}
|abc| &> N^3_{(a,b,c)}\exp\left((12-\delta')\tfrac{(\log\log N_{(a,b,c)})^{\frac{1}{2}}}{\log\log\log N_{(a,b,c)}}\right) \\
&> N^3_{(a,b,c)}\exp\left((\log\log N_{(a,b,c)})^{\frac{1}{2}-\gamma}\right),
\end{aligned}$$

as desired. □

5. **Appendix: Proof of Theorem 2.2**

First, for ease of reference, we review the statement of Theorem 2.2:

> *Let $N_0, \gamma \in \mathbb{R}_{>0}$ be such that $\gamma < \frac{1}{2}$. Then there exists an abc-triple (a,b,c) such that*
>
> $$N_{(a,b,c)} > N_0,$$
> $$|abc| > N^3_{(a,b,c)}\exp\left((\log N_{(a,b,c)})^{\frac{1}{2}-\gamma}\right).$$

Next, we introduce notation as follows:

- Let $\delta \in \mathbb{R}_{>0}$ be such that
$$\delta < 12.$$
Then observe that there exists a $\delta' \in \mathbb{R}_{>0}$ such that
$$\delta' < 12, \quad \tfrac{12-\delta}{(1+\delta)^{\frac{1}{2}}} > 12 - \delta'.$$

- We define $q \in \mathfrak{Primes}$ to be the smallest odd prime number such that
$$q > N_0.$$
Write $w \in \mathbb{V}$ for the q-adic valuation on $\mathbb{Q}$.
- In the following discussion, we shall construct an element
$$x_0 \in \mathbb{R}_{>3}$$
which depends only on N_0 and δ. Note that q depends only on N_0. Let
$$x \in \mathbb{R}_{>x_0}.$$
Write
$$y := (\log x)^{\frac{1}{2}}.$$
We define $G \in \mathbb{Z}_{\geq 1}$ to be the smallest positive integer such that
$$G > \log x.$$
Thus, for a suitable choice of x_0, it follows from Theorem 3.9 (where we take γ to be $\frac{1}{2}$) that
$$\Psi(x, y; q) = \exp\left((\log x)^{\frac{1}{2}} + 4\tfrac{(\log x)^{\frac{1}{2}}}{\log\log x} + O\left(\tfrac{(\log x)^{\frac{1}{2}}}{(\log\log x)^2}\right)\right). \qquad (\ddagger_1)$$

- Observe that there exists a unique $I \in \mathbb{Z}$ such that
$$\tfrac{1}{q}\Psi(x, y; q) \leq Gq^I < \Psi(x, y; q). \qquad (\ddagger_2)$$
It follows immediately from the estimate $(\ddagger_1)$ that, for a suitable choice of x_0, we may suppose that $I \geq 1$.

Lemma 5.1. *For a suitable choice of x_0, there exists a pair of positive integers (a_1, b_1) such that*

(1) $$\mathrm{LPN}(a_1) \leq y,\ \mathrm{LPN}(b_1) \leq y,$$
(2) $$(a_1, b_1) = 1,\ (a_1, q) = 1,\ (b_1, q) = 1,$$
(3) $$1 \leq a_1 \leq x,\ 1 \leq b_1 \leq x,$$
(4) $$1 < \tfrac{b_1}{a_1} < 3,$$
(5) $$\|\tfrac{b_1}{a_1} - 1\|_w \leq \tfrac{1}{q^I}.$$

Proof. First, let us recall the estimate $(\ddagger_2)$
$$Gq^I < \Psi(x, y; q).$$

Thus, by considering the residue classes modulo q^I of the set of integers that appears in the definition of $\Psi(x, y; q)$, we conclude from the Box Principle that there exists a sequence of $G + 1$ integers $2 \leq s_0 < \cdots < s_G \leq x$ such that

$$\mathrm{LPN}(s_i) \leq y \text{ for } i = 0, \ldots, G,$$
$$(s_i, q) = 1 \text{ for } i = 0, \ldots, G,$$
$$s_i \equiv s_j \bmod q^I \text{ for } i, j = 0, \ldots, G.$$

Next, let us suppose that $s_{i+1} > x^{\frac{1}{\log x}} \cdot s_i$ for $i = 0, \ldots, G-1$. Since $G > \log x$, it follows immediately that

$$x \geq s_G > x^{\frac{1}{\log x}} \cdot s_{G-1} > \cdots > x^{\frac{G}{\log x}} \cdot s_0 > xs_0 \geq x$$

— a contradiction. Thus, there exists an $i_0 \in \mathbb{Z}$ such that

$$0 \leq i_0 \leq G - 1,$$
$$s_{i_0} < s_{i_0+1} \leq x^{\frac{1}{\log x}} s_{i_0}.$$

Since $x^{\frac{1}{\log x}} = \exp(1) < 3$, it follows that

$$s_{i_0} < s_{i_0+1} < 3s_{i_0}.$$

We define $a_1, b_1 \in \mathbb{Z}_{\geq 1}$ as follows:

$$(a_1, b_1) = 1,$$
$$\tfrac{b_1}{a_1} := \tfrac{s_{i_0+1}}{s_{i_0}}.$$

Then it follows immediately from the definition of (a_1, b_1) that (a_1, b_1) satisfies conditions (1), (2), (3), (4), and (5) of Lemma 5.1. This completes the proof. $\square$

Lemma 5.2. *There exists an abc-triple* (a, b, c) *such that*

$$N_{(a,b,c)} > N_0, \tag{1}$$

$$|abc| > N_{(a,b,c)}^3 \exp\left((12 - \delta') \frac{(\log N_{(a,b,c)})^{\frac{1}{2}}}{\log\log N_{(a,b,c)}}\right). \tag{2}$$

Proof. It follows from Lemma 5.1 that, for a suitable choice of x_0, there exists a pair of positive integers (a_1, b_1) which satisfies the conditions of Lemma 5.1.

Let

$$a := a_1,\ b := -b_1,\ c := -a_1 + b_1.$$

Since $\frac{b_1}{a_1} > 1$ (cf. condition (4) of Lemma 5.1), $c \neq 0$. Thus, it follows from condition (2) of Lemma 5.1 that (a, b, c) is an *abc*-triple. Next, observe that, since $I \geq 1$, it follows from conditions (2) and (5) of Lemma 5.1 that $q \mid c$. Since $q \mid c$ and $N_0 < q \in \mathfrak{Primes}$, it follows that (a, b, c) satisfies condition (1) of Lemma 5.2, i.e., $N_{(a,b,c)} > N_0$. Thus, it suffices to show that (a, b, c) satisfies condition (2) of Lemma 5.2.

First, since $\mathrm{LPN}(a_1) \leq y$, $\mathrm{LPN}(b_1) \leq y$ (cf. condition (1) of Lemma 5.1), it follows that

$$\prod_{\substack{p \in \mathfrak{Primes}, \\ p \mid ab}} p \leq \prod_{\substack{p \in \mathfrak{Primes}, \\ p \leq y}} p = \exp(\theta(y)).$$

Next, since $(a_1, q) = 1$ and $\|\frac{b_1}{a_1} - 1\|_w \leq \frac{1}{q^I}$ (cf. conditions (2) and (5) of Lemma 5.1), and $I \geq 1$, it follows that

$$q^I \mid c.$$

Thus, it follows from the definition of $N_{(a,b,c)}$ that

$$N_{(a,b,c)} \leq \left(\prod_{\substack{p \in \mathfrak{Primes}, \\ p \mid ab}} p \right) \left(\prod_{\substack{p \in \mathfrak{Primes}, \\ p \mid c}} p \right) \leq \exp(\theta(y)) \cdot \frac{|c|}{q^{I-1}}.$$

Next, since the positive integers a_1 and b_1 satisfy the inequalities $\frac{b_1}{a_1} > 1$ and $\frac{b_1}{a_1} < 3$ (cf. condition (4) of Lemma 5.1), it follows immediately that

$$|b| > |c|, \; |a| > \tfrac{1}{3}|b|.$$

Thus, it follows that

$$|abc| \geq \tfrac{1}{3}|c|^3 \geq \tfrac{1}{3q^3} \left(N_{(a,b,c)} \exp(-\theta(y)) q^I \right)^3 = C_1 \left(N_{(a,b,c)} \exp(-\theta(y)) q^I \right)^3,$$

where we write $C_1 := \frac{1}{3q^3}$. Next, since $\log x > \log x_0 > \log 3 > 1$, it follows from the inequality ($\ddagger_2$) that

$$q^I \geq \tfrac{\Psi(x,y;q)}{Gq} \geq \tfrac{\log x}{(\log x + 1)q} \tfrac{\Psi(x,y;q)}{\log x} \geq \tfrac{1}{2q} \tfrac{\Psi(x,y;q)}{\log x} = C_2 \tfrac{\Psi(x,y;q)}{\log x},$$

where we write $C_2 := \frac{1}{2q}$. Thus, it follows that

$$|abc| \geq C_1 C_2^3 \left(\tfrac{N_{(a,b,c)} \exp(-\theta(y)) \Psi(x,y;q)}{\log x} \right)^3 = C_3 \left(\tfrac{N_{(a,b,c)} \exp(-\theta(y)) \Psi(x,y;q)}{\log x} \right)^3,$$

where we write $C_3 := C_1 C_2^3$. Note that C_3 depends only on N_0. Thus, it follows from Corollary 3.4 and the estimate ($\ddagger_1$) that

$$\begin{aligned}
&\tfrac{\exp(-\theta(y))\Psi(x,y;q)}{\log x} \\
= \;& \exp\left(-(\log x)^{\frac{1}{2}} + O\left(\tfrac{(\log x)^{\frac{1}{2}}}{(\log\log x)^2} \right) \right) \times \\
& \exp\left((\log x)^{\frac{1}{2}} + 4\tfrac{(\log x)^{\frac{1}{2}}}{\log\log x} + O\left(\tfrac{(\log x)^{\frac{1}{2}}}{(\log\log x)^2} \right) \right) \times \\
& \exp(-\log\log x) \\
= \;& \exp\left(4\tfrac{(\log x)^{\frac{1}{2}}}{\log\log x} + O\left(\tfrac{(\log x)^{\frac{1}{2}}}{(\log\log x)^2} \right) \right),
\end{aligned}$$

and hence, for a suitable choice of x_0, that

$$|abc| > N_{(a,b,c)}^3 \exp\left((12 - \delta) \tfrac{(\log x)^{\frac{1}{2}}}{\log\log x} \right).$$

Finally, let us estimate $N_{(a,b,c)}$. First, it follows from the estimate of $N_{(a,b,c)}$ obtained above, together with the definition of c and the inequality $\frac{b_1}{a_1} > 1$, that

$$N_{(a,b,c)} \leq \exp(\theta(y)) \cdot |c| \leq \exp(\theta(y)) \cdot |b|.$$

Next, for a suitable choice of x_0, it follows from Corollary 3.4 and condition (3) of Lemma 5.1 that

$$\exp(\theta(y)) \leq \exp(2(\log x)^{\frac{1}{2}}),$$
$$|b| = b_1 \leq x.$$

Next, for a suitable choice of x_0, it follows from an elementary calculation that

$$\exp\left(2(\log x)^{\frac{1}{2}}\right) \leq x^{\delta}.$$

Thus, we conclude that

$$N_{(a,b,c)} \leq x^{1+\delta},$$

i.e.,

$$\log N_{(a,b,c)} \leq (1+\delta)\log x.$$

Next, observe that we may assume without loss of generality that

$$\log N_0 > \exp(2).$$

Since the function

$$z \mapsto \tfrac{z^{\frac{1}{2}}}{\log z} \text{ for } z \in \mathbb{R}_{>\exp(2)}$$

is strictly monotone increasing, $\frac{12-\delta}{(1+\delta)^{\frac{1}{2}}} > 12-\delta'$, and $N_{(a,b,c)} > N_0$, it follows that, for a suitable choice of x_0,

$$\begin{aligned}
& \exp\left((12-\delta)\tfrac{(\log x)^{\frac{1}{2}}}{\log\log x}\right) \\
= \;& \exp\left(\tfrac{12-\delta}{(1+\delta)^{\frac{1}{2}}} \cdot \tfrac{\log\log x + \log(1+\delta)}{\log\log x} \cdot \tfrac{((1+\delta)\log x)^{\frac{1}{2}}}{\log((1+\delta)\log x)}\right) \\
> \;& \exp\left((12-\delta')\tfrac{((1+\delta)\log x)^{\frac{1}{2}}}{\log((1+\delta)\log x)}\right) \\
\geq \;& \exp\left((12-\delta')\tfrac{(\log N_{(a,b,c)})^{\frac{1}{2}}}{\log\log N_{(a,b,c)}}\right).
\end{aligned}$$

Thus, it follows that (a,b,c) satisfies condition (2) of Lemma 5.2, i.e.,

$$|abc| > N_{(a,b,c)}^3 \exp\left((12-\delta')\tfrac{(\log N_{(a,b,c)})^{\frac{1}{2}}}{\log\log N_{(a,b,c)}}\right).$$

This completes the proof. $\square$

Proof of Theorem 2.2. Observe that there exists an $M \in \mathbb{R}_{>0}$ such that, for $z \in \mathbb{R}_{>M}$,

$$\tfrac{\log z}{12-\delta'} < z^{\gamma}.$$

Now we apply Lemma 5.2. Observe that we may assume without loss of generality that

$$\log N_0 > M.$$

Since $N_{(a,b,c)} > N_0$, it follows that

$$\tfrac{\log\log N_{(a,b,c)}}{12-\delta'} < (\log N_{(a,b,c)})^{\gamma}.$$

Thus, we conclude that

$$\begin{aligned}
|abc| &> N_{(a,b,c)}^3 \exp\left((12-\delta')\tfrac{(\log N_{(a,b,c)})^{\frac{1}{2}}}{\log\log N_{(a,b,c)}}\right) \\
&> N_{(a,b,c)}^3 \exp\left((\log N_{(a,b,c)})^{\frac{1}{2}-\gamma}\right),
\end{aligned}$$

as desired. $\square$

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